ON THE NUMERICAL SOLUTION OF THE PLANE ELASTO-PLASTIC PROBLEM

by

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1. *Introduction.*

In a previous paper [1] a method of solving the plane elasto-plastic problem was described. The point was to determine the contour C that separates the plastic from the elastic region such that the stresses, taking assigned values on some closed curve B and at infinity, are continuous functions throughout the exterior of the boundary B. Analytical methods for the solution of this boundary-value problem can be applied in some special cases; generally however only numerical methods lead to the required results. By means of an illustrative example we will go into the numerical treatment of the relations which can be used for the determination of the contour C.

For the elastic region we have the basic Kolosov-Muskhelishvili equations

$$\sigma_x + \sigma_y = 2 \left[\varphi(z) + \overline{\varphi(z)} \right] \tag{1}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2 \left[\bar{z} \frac{d\varphi}{dz} + \psi(z) \right] \tag{2}$$

The functions φ and ψ of the complez variable z have the form

$$\varphi(z) = - \frac{X + iY}{2\pi(1+\kappa)} \frac{1}{z} + \left[\frac{p+q}{4} + ic \right] + \varphi_o(z) \tag{3}$$

$$\psi(z) = \frac{\kappa(X - iY)}{2\pi(1+\kappa)} \frac{1}{z} + \left[- \frac{p-q}{2} + it \right] + \psi_o(z), \tag{4}$$

where (X, Y) is the resultant vector of the external forces applied to the boundary B, $p=\sigma_x(\infty)$, $q=\sigma_y(\infty)$, $t=\tau_{xy}(\infty)$ and the functions φ_o and ψ_o are represented by the series

$$\varphi_o(z) = \sum_{k=2}^{\infty} \frac{\alpha_k}{z^k} \tag{5}$$

and

$$\psi_o(z) = \sum_{k=2}^{\infty} \frac{\beta_k}{z^k}, \tag{6}$$

where the origin of the complex z-plane is assumed to lie in the interior of B. If φ denotes the angle between the direction of the largest of the two principal stresses σ_1, σ_2 and the x-axis at some point (x, y) then

$$\sigma_x = \sigma + \rho \cos 2\varphi \tag{7}$$

$$\sigma_y = \sigma - \rho \cos 2\varphi \tag{8}$$

$$\tau_{xy} = \rho \sin 2\varphi, \tag{9}$$

where for brevity we have put

$$\sigma = \tfrac{1}{2} (\sigma_1 + \sigma_2)$$

$$\rho = \tfrac{1}{2} (\sigma_1 - \sigma_2).$$

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In the plastic region some plasticity-condition defining ρ as a function of σ must be satisfied. This condition yields two hyperbolic differential equations for the quantities σ and φ. For the characteristics we have the differential equation

$$\frac{dy}{dx} = \operatorname{tg}\mu, \tag{10}$$

where $\mu = \varphi \pm \frac{\alpha}{2}$, α denoting the supplement of the angle between the normal at the envelope of the circles with radius $\rho(\sigma)$ and the σ-axis in Mohr's diagram. The characteristic equations have the simple form:

$$\frac{\partial\Phi}{\partial\mu_1} + \frac{\partial\varphi}{\partial\mu_1} = 0 \tag{11}$$

$$\frac{\partial\Phi}{\partial\mu_2} - \frac{\partial\varphi}{\partial\mu_2} = 0, \tag{12}$$

where

$$\Phi(\sigma) = \int \frac{\sin\alpha}{2\rho} d\sigma. \tag{13}$$

2. *The numerical procedure.*

Let

$$z = \omega(\zeta) = \frac{c}{\zeta} + \sum_{k=0}^{\infty} \gamma_k \zeta^k \tag{14}$$

be a conformal mapping that maps the exterior of the (unknown) contour C in the z-plane onto the interior of the unit-circle in the complex ζ-plane such that $\omega(o) = \infty$. Now the left members of the Kolosov-Muskhelishvili equations are taken as to be determined from the assigned values on the boundary B by the method of characteristics in the plastic region whereas the functions φ_0 and ψ_0 refer to the elastic region. In this sense the equations (1) and (2) are only valid on the contour C. Thus the point is to determine a mapping (14) such that both members of the equations (1) and (2) are identical on the unit-circle in the ζ-plane.
Suppose the left members of the equations (1) and (2) to be mapped on the Fourier-series

$$\sigma_x + \sigma_y \rightarrow F(\sigma) = \sum_{n=-\infty}^{\infty} c_n \sigma^n \tag{15}$$

and

$$\sigma_y - \sigma_x - 2i\tau_{xy} \rightarrow G(\sigma) = \sum_{n=-\infty}^{\infty} d_n \sigma^n, \tag{16}$$

where $\sigma = e^{i\theta}$. If in the Kolosov-Muskhelishvili equations

$$\varphi(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k \tag{17}$$

and

$$\psi(\zeta) = \sum_{k=0}^{\infty} b_k \zeta^k, \tag{18}$$

then we have the following relations for the coefficients in the expansion of the function $\omega(\zeta)$:

$$f_k = \sum_{m=1}^{\infty} m\bar{a}_m \gamma_{m+k-1} + \lambda_{k-2}\, c - \sum_{m=1}^{\infty} m\lambda_{m+k-1}\bar{\gamma}_m = 0, \quad k > 0, \tag{19}$$

where

$$2a_n = c_n, \quad n \geqslant 1$$

$$2\lambda_n = d_n, \quad n \geqslant 1$$

$$2\lambda_o = d_o + p - q + 2it$$

$$2\lambda_{-1} = d_{-1} + 2\kappa a_1.$$

Moreover we have the boundary-condition

$$c_o = p + q. \tag{20}$$

Assuming central symmetry we have that the coefficients $\gamma_0, \gamma_2, \gamma_4, \ldots\ldots\ldots$; $a_1, a_3, a_5, \ldots\ldots$ and $\lambda_{-1}, \lambda_1, \lambda_3, \lambda_5, \ldots\ldots$ vanish. In case of symmetry with respect to the x-axis in the z-plane the coefficients γ_k are real. Central symmetry and symmetry with respect to the x-axis imply symmetry with respect to the y-axis.
In order to solve the problem terms of order higher than some n in the expansion of the function $\omega(\zeta)$ are neglected. To some given initial set of variables $c, \gamma_0, \gamma_1, \ldots\ldots, \gamma_n$ corresponds a transformation-function $\omega_1(\zeta)$ that maps a curve C_1 onto the unit-circle in the complex ζ-plane. Application of the equations (11) and (12) yields stresses in the region bounded by C_1, especially on the contour C_1. The mapping-function ω_1 defines therefore the stresses σ_x, σ_y and τ_{xy} on the unit-circle in the complex ζ-plane. Thus from an expansion of the functions $\sigma_x + \sigma_y$ and $\sigma_y - \sigma_x - 2i\tau_{xy}$ in a Fourier-series $\Sigma c_k \sigma^k$ and $\Sigma d_k \sigma^k$ coefficients c_k and d_k are available so that the functions f_k defined by (19) can be calculated. For the solution of the problem the functions f_k must vanish. Therefore one can find this solution by minimizing the function

$$S = \Sigma f_k \bar{f}_k \tag{21}$$

subject to the condition (20); starting from the initial point on the surface S one has to change the variables $c, \gamma_0, \gamma_1, \ldots\ldots, \gamma_n$ such that S decreases. Repetition of this process will eventually lead to some point from which no decrease of S is possible wthin the limits of accuracy assigned.
Let for example the boundary B an ellipse with major axis 2a and minor axis 2b and let for simplicity the plasticity-condition be

$$\rho = k, \tag{22}$$

where k is a constant. At the boundary B to which a normal loading f is applied we have the stresses

$$\sigma_1 = 2k + f$$

$$\sigma_2 = f$$

and therefore

$$\sigma = k + f. \tag{23}$$

From (10) we have for the characteristics in the plastic region the differential equation

$$\frac{dy}{dx} = \mathrm{tg}\mu = \mathrm{tg}\ (\varphi \pm \frac{\alpha}{2}) = \mathrm{tg}\ (\varphi \pm \frac{\pi}{4}) \tag{24}$$

The characteristics starting from the points

$$\left.\begin{aligned} x &= a \cos \vartheta = a \cos k\Delta\vartheta \\ y &= b \sin \vartheta = b \sin k\Delta\vartheta \end{aligned}\right\} , k = 0, 1, 2, \ldots\ldots \quad (0 \leqslant \vartheta < 2\pi) \tag{25}$$

of the ellipse constitute an orthogonal net. Along these characteristics we have the characteristic equations

$$\frac{\partial}{\partial\mu_1}(\sigma + 2\rho\varphi) = 0 \tag{26}$$

$$\frac{\partial}{\partial\mu_2}(-\sigma + 2\rho\varphi) = 0 \tag{27}$$

where

$$\mu_1 = \varphi + \frac{\pi}{4}$$

$$\mu_2 = \varphi - \frac{\pi}{4}.$$

The function $\vartheta_1 = \sigma + 2\rho\varphi$ does not change in a characteristic direction μ_1 and the function $\vartheta_2 = -\sigma + 2\rho\varphi$ does not change in a characteristic direction μ_2. Therefore in some net-point where two characteristics intersect we have for the quantities φ and σ the relations

$$\varphi = \frac{\vartheta_1 + \vartheta_2}{4\rho} \tag{28}$$

$$\sigma = \frac{\vartheta_1 - \vartheta_2}{2}. \tag{29}$$

From the values of φ and σ in these net-points we can evaluate φ and σ at the nodes (x_i, y_j) of a square mesh which is constituted by equidistant lines parallel to the x- and y-axes. If φ_{ij} and σ_{ij} denote the values of φ and σ at these nodes then we can form the matrices Φ and Σ with elements φ_{ij} and σ_{ij} respectively. From these matrices we find φ and σ at some point (x, y) in the plastic region by quadratic interpolation; for

$$x_{i-1} \leqslant x \leqslant x$$

and

$$y_{j-1} \leqslant y \leqslant y_j$$

we find $\varphi(x,y)$ from the interpolation-formula

$$\varphi(x,y) = \frac{1}{h^2}\Big[\varphi_{ij}(x-x_{i-1})(y-y_{j-1}) + \varphi_{i-1,j-1}(x-x_i)(y-y_j) - \varphi_{i,j-1}(x-x_{i-1})(y-y_j) - \varphi_{i-1,j}(x-x_i)(y-y_{j-1})\Big] \tag{30}$$

and $\sigma(x,y)$ from the formula

$$\sigma(x,y) = \frac{1}{h^2}\Big[\sigma_{ij}(x-x_{i-1})(y-y_{j-1}) + \sigma_{i-1,j-1}(x-x_i)(y-y_j) - \sigma_{i,j-1}(x-x_{i-1})(y-y_j) - \sigma_{i-1,j}(x-x_i)(y-y_{j-1})\Big], \tag{31}$$

where h is the size of the square mesh. It is observed that to some given plasticity-condition and prescribed boundary-values of φ and σ at B there correspond fixed matrices Φ and Σ.
From (15) we find that

$$\sigma_x + \sigma_y = c_o + \sum_{k=1}^{\infty} c_k\, e^{ik\theta} + \sum_{k=1}^{\infty} c_{-k}\, e^{-ik\theta}$$

$$= c_o + \sum_{k=1}^{\infty} c_k\,(\cos k\theta + i\sin k\theta) + \sum_{k=1}^{\infty} \bar{c}_k\,(\cos k\theta - i\sin k\theta).$$

Because of the central symmetry and the symmetry with respect to the axes the coefficients $c_1, c_3, c_5, \ldots\ldots$ vanish and $\bar{c}_{2k} = c_{2k}$. Therefore

$$\sigma_x + \sigma_y = c_o + 2\sum_{k=1}^{\infty} c_{2k}\cos 2k\theta. \tag{32}$$

As for (16) we deduce from symmetry-considerations that the coefficients d_{2k+1} vanish and that the remaining coefficients d_{2k} are real so that

$$\sigma_y - \sigma_x = d_o + \sum_{k=1}^{\infty}(d_{2k} + d_{-2k})\cos 2k\theta. \tag{33}$$

and

$$-2\tau_{xy} = \sum_{k=1}^{\infty}(d_{2k} - d_{-2k})\sin 2k\theta. \tag{34}$$

From (30) and (31) we find for certain values of the variables $c, \gamma_1, \gamma_2, \ldots\ldots, \gamma_n$ the quantities φ and σ at N points $\sigma_j = e^{i\theta_j}$, where

$$\theta_j = j.\frac{2\pi}{N},\quad j = 0, 1, 2, \ldots\ldots, N-1 \tag{35}$$

on the unit-circle in the ζ-plane. From (7), (8) and (9) we then obtain the left members of the relations (32), (33) and (34) at the points σ_j. Thus from the relations

$$c_o = \frac{1}{N}\sum_{j=0}^{N-1}(\sigma_x + \sigma_y)_j$$

$$c_{2k} = \frac{1}{N}\sum_{j=0}^{N-1}(\sigma_x + \sigma_y)_j \cos 2k\theta_j$$

$$d_o = \frac{1}{N}\sum_{j=0}^{N-1}(\sigma_y - \sigma_x)_j$$

$$d_{2k} = \frac{1}{N}\sum_{j=0}^{N-1}\left[(\sigma_y - \sigma_x)_j \cos 2k\theta_j - 2(\tau_{xy})_j \sin 2k\theta_j\right]$$

coefficients c_k and d_k are available and the function S can be calculated. As for the condition (20) one may add the term $f\bar{f}$ to the sum S, where

$$f = c_o - p - q.$$

It is observed that making use of symmetry-properties of the functions involved one can restrict himself to only the first quadrant in the z-plane and moreover effect considerable improvements in the calculation-procedure [2] .

As regards the minimization of the function S there are several optimization-techniques available. Application of some gradient-technique requires the existence of the gradient-vector ∇S. However generally S is not differentiable and therefore one might better apply an optimizationtechnique like Rosenbrock's method [3] .

Finally it is remarked that the rate of convergence strongly depends on the initial set of variables $c, \gamma_1, \gamma_3, \ldots, \gamma_n$ from which the process is started. Now if the boundary B is a circle with radius R, the contour C is an ellipse

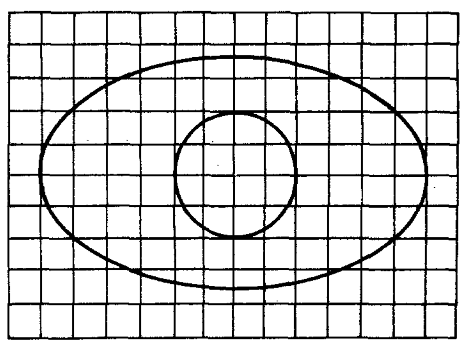
$$z = c\left\{\frac{1}{\zeta} + \beta\zeta\right\},$$

where

$$c = Re^{\frac{1}{2k}\left[\frac{p+q}{2}-f-k\right]}$$

and

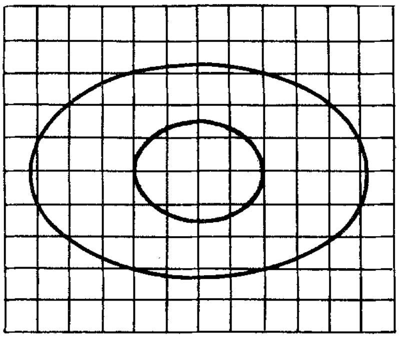
$$\beta = \frac{q - p}{2k}.$$



$f = 0$, $p = 2\frac{1}{2}$, $q = 3$

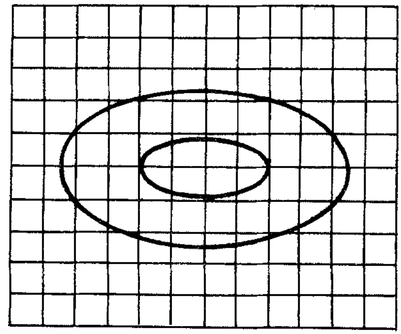
Fig. 1.

Therefore if the boundary B is an ellipse with semi-axes R and R - ΔR this function may be taken as a first approximation $\omega_1(\zeta)$ of the mapping-function $z = \omega(\zeta)$ and thus the ellipse with axes $2c(1+\beta)$ and $2c(1-\beta)$ is a first approximation C_1 of the contour C. From this initial approximation the minimization of the function S should be started. Then the solution of this problem may again be used as a first approximation for the next problem, the boundary B being an ellipse with semi-axes R and R - 2ΔR. Going on in this way one starts the minimization of the function S, the boundary B being an ellipse with axes 2a and 2b from the solution of the preceding problem where the boundary B is an ellipse with axes 2a and 2(b+Δb); the start of the



$f = 0,$ $p = 2\frac{1}{2}$ $q = 3$

Fig. 2.



$f = 0,$ $p = 2\frac{1}{2}$ $q = 3$

Fig. 3.

next calculation is the solution of the preceding one.

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